MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2016

Calculator-assumed

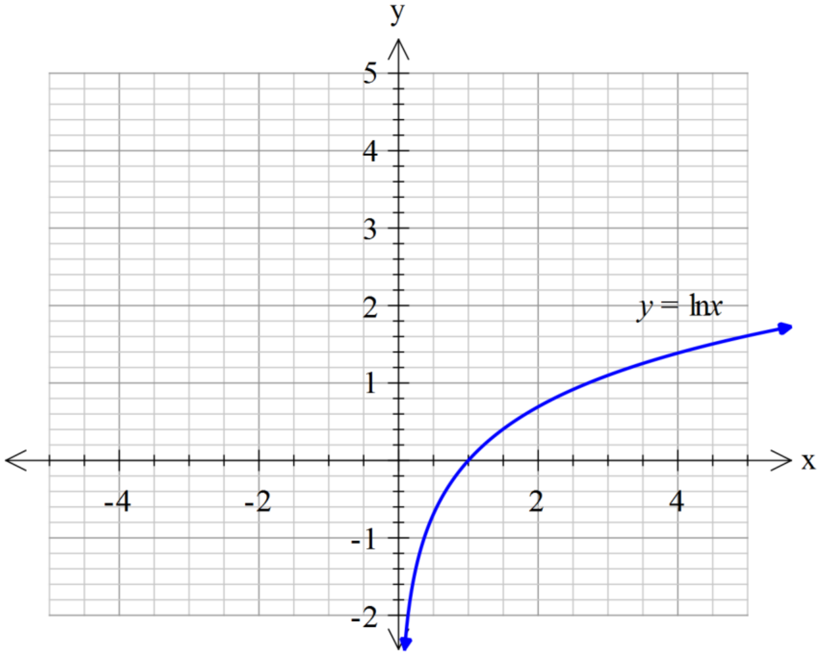
Marking Key

The release date for this exam and marking scheme is

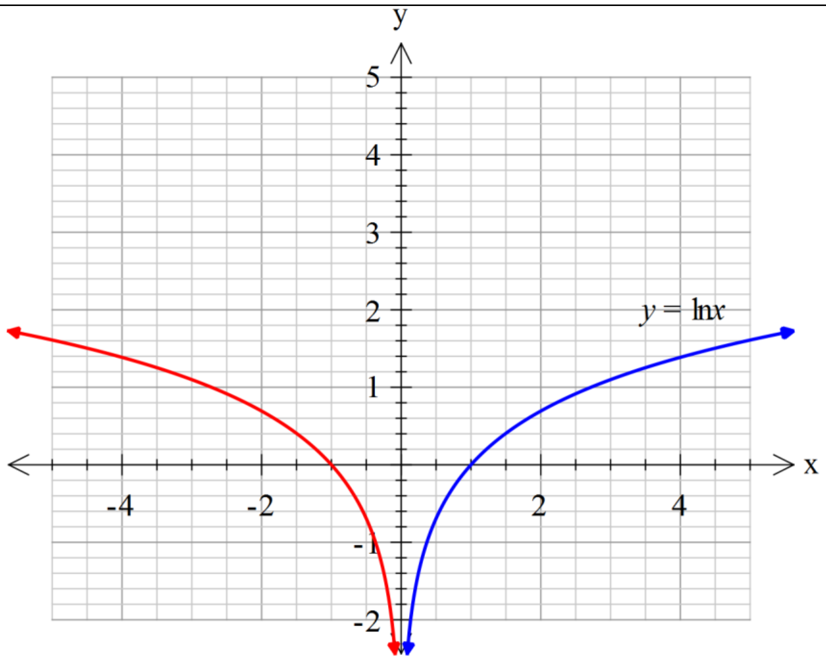
- **the end of week 1 of term 4, 2016**

Question 8 **(4 marks)**

Consider the graph of $y = \ln x,\ \ x > 0$ as shown below.



(a) Sketch the function $y = \ln(-x),\quad x < 0$ on the axes above. (2 marks)

Solution

Specific behaviours
✓ shows a reflection in the y axis ✓ shows x intercept

(b) Determine the gradient function for $y = \ln(-x), \quad x < 0$. (2 marks)

Solution
$y = \ln(-x), \quad x < 0$ $\frac{dy}{dx} = \frac{-1}{-x} = \frac{1}{x}$
Specific behaviours
✓ uses chain rule ✓ determines derivative

Question 9 **(7 marks)**

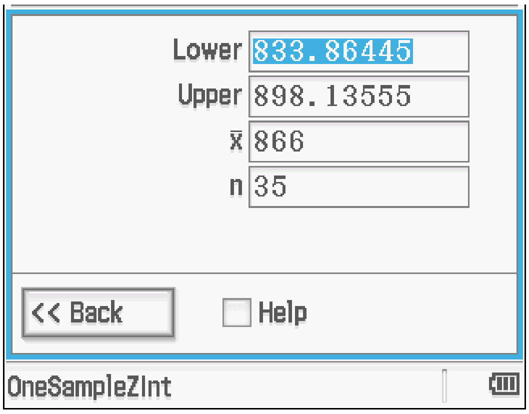
A certain type of battery is being considered by a car manufacturer for a new electric car. The manufacturer collected a sample of 35 such batteries and found that the mean life span was 866 days with a sample standard deviation of 97 days.

(a) Assuming a normal distribution for the sample means determine a 95% confidence interval for the population mean life span of the battery. (3 marks)

Solution

$$\bar{x} - 1.960\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1.960\frac{\sigma}{\sqrt{n}}$$

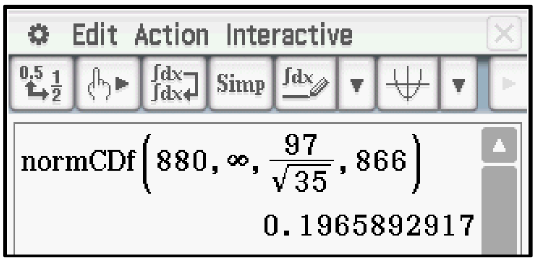
$833.86 \le \mu \le 898.14$



Specific behaviours

✓ uses correct z score
✓ determines upper limit
✓ determines lower limit

(b) If the manufacturer collected a new sample of 35 such batteries, determine the probability that the mean life span will be greater than 880 days. (2 marks)

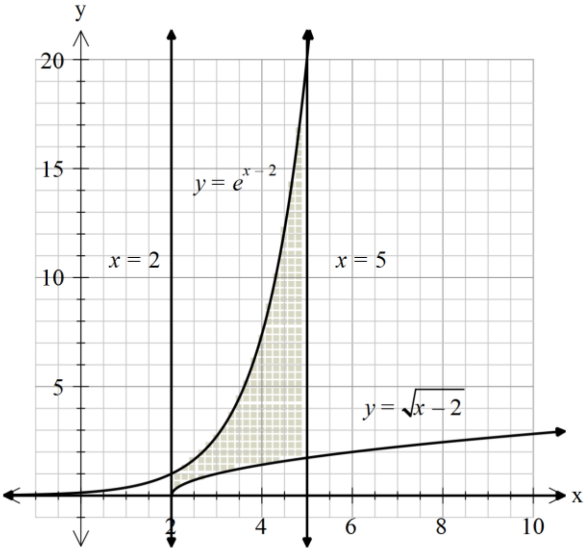
Solution
$P(\bar{X} > 880) \approx 0.19659$ 
Specific behaviours
✓ uses normal distribution ✓ determines probability

(c) The manufacturers are looking for a battery to last 900 days. Would you advise that they use this type of battery? Justify your answer. (2 marks)

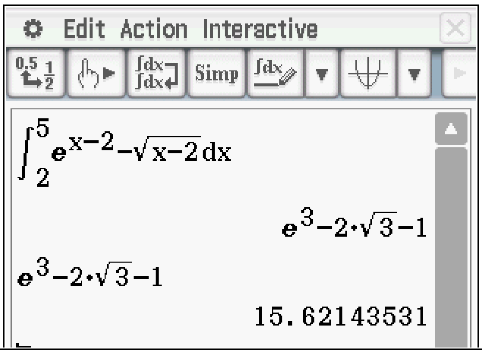
Solution
No, as this lies outside the confidence interval.
Specific behaviours
✓ Answers No ✓ uses confidence interval in explanation

Question 10 **(8 marks)**

Consider the area between the curves $y=\sqrt{x-2}$, $y=e^{x-2}$ and the lines $x=5$ and $x=2$ as shown below.

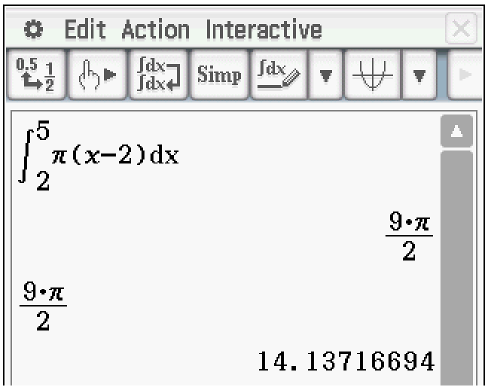


(a) State an appropriate integral expression for the area shown and evaluate it. (2 marks)

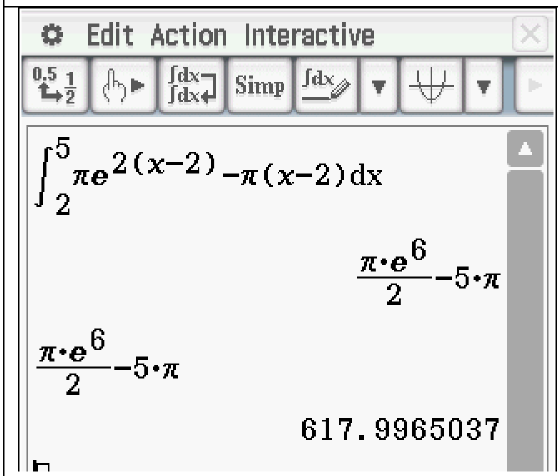
Solution

Specific behaviours
✓ states an appropriate integral to determine the area with correct limits ✓ determines area (either exactly or in as a decimal approximation)

The area shown above is rotated about the x axis forming a three-dimensional object with a cavity (space) that has capacity.

(b) Determine the capacity of this object. (3 marks)

Solution

Specific behaviours
✓ states solid of rotation integral ✓ uses correct limits ✓ determines capacity

(c) Determine the volume of the walls of this object. (3 marks)

Solution

Specific behaviours
✓ uses subtraction of solids of rotation ✓ uses correct limits ✓ determines volume of walls

Question 11 **(7 marks)**

A complex number z, is defined by $|z| = \sqrt{2}$ and $arg\ z = \frac{\pi}{6}$.

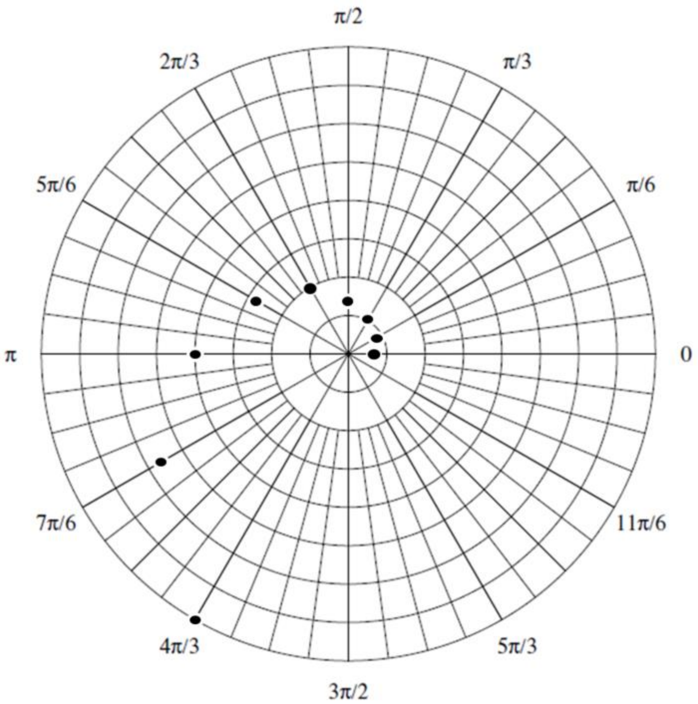
(a) On the polar grid below, graph the sequence z^n for integers, $0 \le n \le 8$. (4 marks)

Solution

Evaluate $r = |z^n|$ and $\theta = arg\left(z^n\right)$ for each value of n within the given domain.

n	0	1	2	3	4	5	6	7	8
$\lvert z^n \rvert$	1	$\sqrt{2}$	2	$2\sqrt{2}$	4	$4\sqrt{2}$	8	$8\sqrt{2}$	16
$arg(z^n)$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$

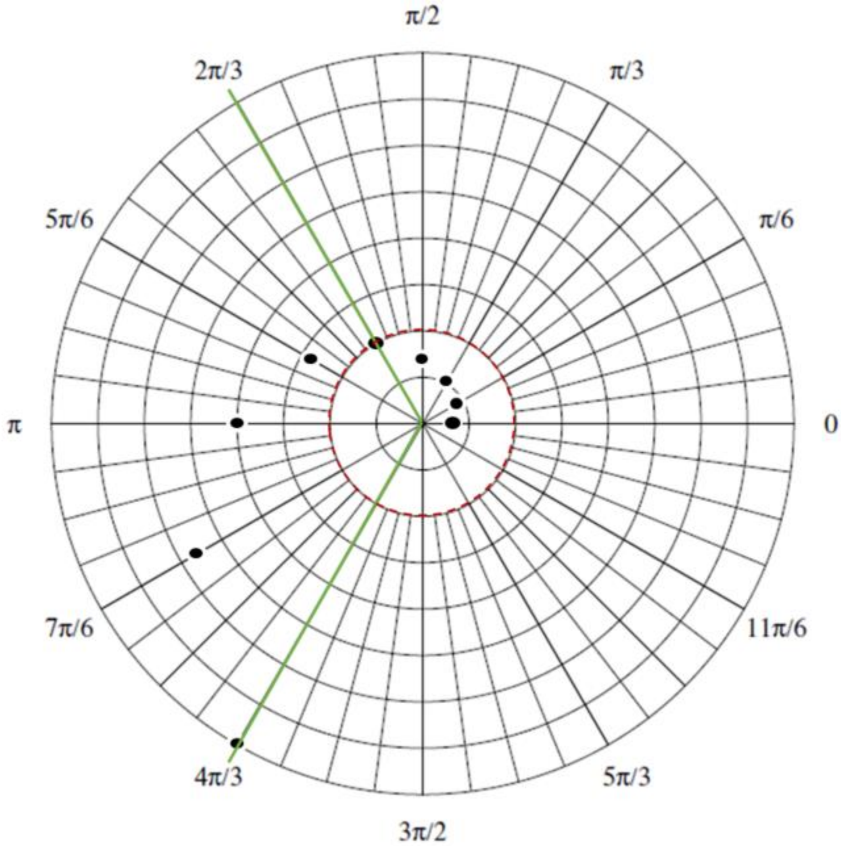
Plotting the points gives …



Specific behaviours

✓ Calculates correct values r of z^n for $0 \le n \le 8$
✓ Calculates correct values θ of z^n for $0 \le n \le 8$
✓ Plots points correctly : mod
✓ Plots points correctly : arg

(b) Hence or otherwise find the value(s) of k, where k is an integer $-6 \le k \le 6$, such that $-\frac{2\pi}{3} < Arg(z^k) < \frac{2\pi}{3}$ and $|z^k| < 4$. (3 marks)

Solution

For $-3 \le k \le -1$, $\left|z^k\right| < 4$ since $\left|z^{-k}\right| = \left|\frac{1}{z^k}\right|$ for $k < 0$ and $-\frac{2\pi}{3} < Arg\left(z\right) < 0$ For $0 \le k \le 3$, $\left|z^k\right| < 4$ and $0 < Arg\left(z\right) < \frac{2\pi}{3}$ Solution: $-3 \le k \le 3$
Specific behaviours
✓ Recognises that $\left|z^k\right| < 4$ for $-3 \le k \le -1$ ✓ Recognises that $\left|z^k\right| < 4$ for $0 \le k \le 3$ ✓ Recognises that $-\frac{2\pi}{3} < Arg(z^k) < \frac{2\pi}{3}$ and $\left|z\right| < 4$ for $-3 \le k \le 3$

Question 12 **(7 marks)**

The graph of the equation $x^2 - 4xy + 4y^2 - 5x + 4 = 0$ is a parabola.

(a) Find the equation of the tangent to the parabola at the point $P(4,0)$. (4 marks)

Solution
Differentiating implicitly: $2x - 4y - 4x\frac{dy}{dx} + 8y\frac{dy}{dx} - 5 = 0$ At $P(4,0)$, $8 - 16\frac{dy}{dx} - 5 = 0$, i.e. $\frac{dy}{dx} = \frac{3}{16}$ Equation of tangent: $y = \frac{3}{16}(x-4)$, i.e. $y = \frac{3}{16}x - \frac{1}{4}$
Specific behaviours
✓✓ implicit differentiation ✓ obtains $\frac{dy}{dx} = \frac{3}{8}$ ✓ obtains equation of tangent

(b) Find the coordinates of the point Q on the parabola where the tangent is parallel to the y –axis. (3 marks)

Solution
Using: $2x - 4y - 4x\frac{dy}{dx} + 8y\frac{dy}{dx} - 5 = 0$ from (a), $\frac{dy}{dx} = \frac{4y - 2x + 5}{8y - 4x}$ Require $8y - 4x = 0$ i.e. $x = 2y$ So $4y^2 - 8y^2 + 4y^2 - 10y + 4 = 0$, i.e. $y = 0.4$ and $x = 0.8$ Q has coordinates (0.8,0.4).
Specific behaviours
✓ obtains $8y - 4x = 0$ ✓ solves for x ✓ states the coordinates of point Q

Question 13 **(10 marks)**

The position vector $\boldsymbol{r}(t)$ of a moving particle P at time t is given by

$$\boldsymbol{r}(t) = (2t+5)\boldsymbol{i} + (2t-7)\boldsymbol{j} + (5-t)\boldsymbol{k}.$$

(a) Describe the path traced out by P. (1 mark)

Solution
The path is a straight line.
Specific behaviours
✓ correct answer

(b) Find the point where the path of P meets the plane whose equation is

$$3x + 4y - 6z = 37.$$

(2 marks)

Solution
Substituting $x = 2t+5, y = 2t-7$ and $z = 5-t$ into the equation of the plane gives $3(2t+5) + 4(2t-7) - 6(5-t) = 37 \Rightarrow 20t - 43 = 37 \Rightarrow t = 4$ Coordinates of the point of intersection are $(x,y,z) = (2\times 4+5, 2\times 4-7, 5-4) = (13,1,1)$
Specific behaviours
✓ obtains t ✓ obtains x, y and z

(c) Find the minimum distance between the path of P and the origin $O(0,0,0)$. (3 marks)

Solution
If ℓ denotes distance from the origin $\ell^2 = (2t+5)^2 + (2t-7)^2 + (5-t)^2$ $= 4t^2 + 20t + 25 + 4t^2 - 28t + 49 + 25 - 10t + t^2$ $= 9t^2 - 18t + 99$ For minimum $t = \frac{18}{2\times 9} = 1$ (using parabola properties, or by differentiation) $\ell^2_{min} = 9 \times 1^2 - 18 \times 1 + 99 = 90$, so minimum distance is $\sqrt{90} \cong 9.487$ (to 3 decimal places)
Specific behaviours
✓ Obtains expression for ℓ^2 ✓ obtains t_{min} ✓ obtains answer

(d) Show that $\boldsymbol{r}(t)$, the position vector of P at time t, and $\boldsymbol{v}(t)$**,** its velocity, are mutually orthogonal when P is closest to the origin. (2 marks)

Solution
$\boldsymbol{r}(1) = (2 \times 1 + 5)\boldsymbol{i} + (2 \times 1 - 7)\boldsymbol{j} + (5 - 1)\boldsymbol{k} = 7\boldsymbol{i} - 5\boldsymbol{j} + 4\boldsymbol{k}$ $\boldsymbol{v}(t) = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ (constant) $\boldsymbol{r}(1) \cdot \boldsymbol{v} = 7 \times 2 - 5 \times 2 + 4 \times -1 = 0 \Longrightarrow$ mutual orthogonality
Specific behaviours
✓ obtains $\mathbf{r}(1)$ ✓ evaluates dot product

The angular momentum of P at time t is the vector $\boldsymbol{H}(t)$ defined by

$$\boldsymbol{H}(t) = m(\boldsymbol{r}(t) \times \boldsymbol{v}(t))$$

where m is the mass of P.

(e) Show that the angular momentum of P is constant. (2 marks)

Solution
$\boldsymbol{H}(t) = m\big(\boldsymbol{r}(t) \times \boldsymbol{v}(t)\big) = m\Big(\big((2t+5)\boldsymbol{i} + (2t-7)\boldsymbol{j} + (5-t)\boldsymbol{k}\big) \times (2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k})\Big)$ $= \mathrm{m}\Big(\big((2\mathrm{t}-7)(-1) - (5-\mathrm{t})2\big)\boldsymbol{i} + \big((5-t)2 - ((2t+5)(-1)\big)\boldsymbol{j} + \big((2t+5)2 -$ $(2t-7)2\big)\boldsymbol{k}\Big)$ $= m(-3\boldsymbol{i} + 15\boldsymbol{j} + 24\boldsymbol{k})$ - a constant vector
Specific behaviours
✓ obtains second line ✓ simplifies

Question 14 **(6 marks)**

(a) Show that the complex number $z = 2+3i$ is a root of

$$P(z) = 2z^3 - 7z^2 + 22z + 13 = 0 .$$ (3 marks)

Solution

Substituting $z = 2+3i$ into $P(z)$ we have:

$$\begin{aligned} P(2+3i) &= 2(2+3i)^3 - 7(2+3i)^2 + 22(2+3i) + 13 \\ &= 2(-46+9i) - 7(-5+12i) + 44 + 66i + 13 \\ &= -92 + 18i + 35 - 84i + 57 + 66i \\ &= 0 \end{aligned}$$

Hence $z = 2+3i$ is a root of $P(z)=0$

Specific behaviours

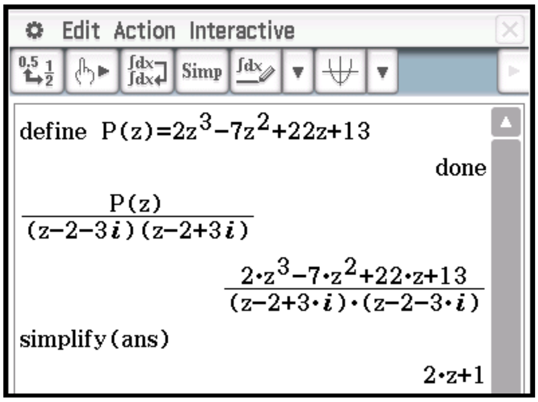
✓ Substitutes into the equation correct
✓ Cubes z value
✓ Simplifies to zero

(b) Using the result from (a) determine all the roots of $P(z) = 0$, justifying your solution and describing the nature of each of the roots. (3 marks)

Solution

$\because z = 2+3i$ is a root, then $\bar{z} = 2-3i$ its complex conjugate, is also a root.

$$\Rightarrow P(z) = (z-2-3i)(z-2+3i)(az+b)$$



$$\begin{aligned} \therefore az+b &= \frac{P(z)}{(z-2-3i)(z-2+3i)} \\ &= \frac{2z^3 - 7z^2 + 22z + 13}{z^2 - 4z + 13} \\ &= 2z+1 \end{aligned}$$

The other root is $z = -\frac{1}{2}$ which is real.

So there are 3 roots, two are complex, namely $z = 2+3i$ and $z = 2-3i$ while the third is $z = -\frac{1}{2}$ which is real.

Specific behaviours

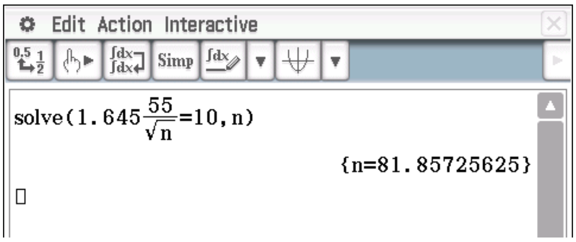
✓ indicates that if $2+3i$ is a root then so is $2-3i$
✓ determines the third factor – showing method (even though may be done on calculator)
✓ states the three roots indicting whether real or complex

Question 15 **(8 marks)**

A quality control unit at a large warehouse will be checking the mean weight of the cans of tomatoes that are stored at the warehouse. The population standard deviation of the weight is 55 grams.

(a) Determine the sample size if we want to be 90% confident that the mean of the sample is within 10 grams of the population mean. (3 marks)

Solution



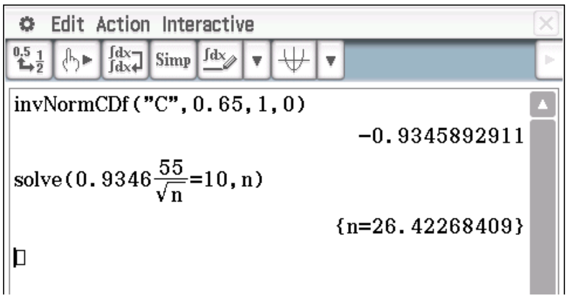
Sample size of 82 cans

Specific behaviours

✓ uses correct z score
✓ solves for sample size
✓ rounds n upward

(b) Determine the sample size if we want to be 65% confident that the mean of the sample is within 10 grams of the population mean. (3 marks)

Solution



Sample size of 27 cans

Specific behaviours

✓ uses correct z score
✓ solves for sample size
✓ rounds n upward

(c) A sample is to be chosen such that we are 99% confident that the sample mean is within 10 grams of the population mean. By what factor should the sample size be changed such that the confidence interval is one third in length but to the same degree of confidence? (2 marks)

Solution
$\frac{1}{3}2.576\frac{55}{\sqrt{n}} = 2.576\frac{55}{\sqrt{9n}}$ Factor of 9
Specific behaviours
✓ uses confidence interval ✓ determines a factor of 9

Question 16 (13 marks)

Let the function f be defined as follows $f(x) = x^2 - 4x + 5$.

(a) Explain why f does not have an inverse over its natural domain. (1 mark)

Solution
The function is a many to one over the natural domain and hence has no inverse.
Specific behaviours
✓ States that function is not one to one function.

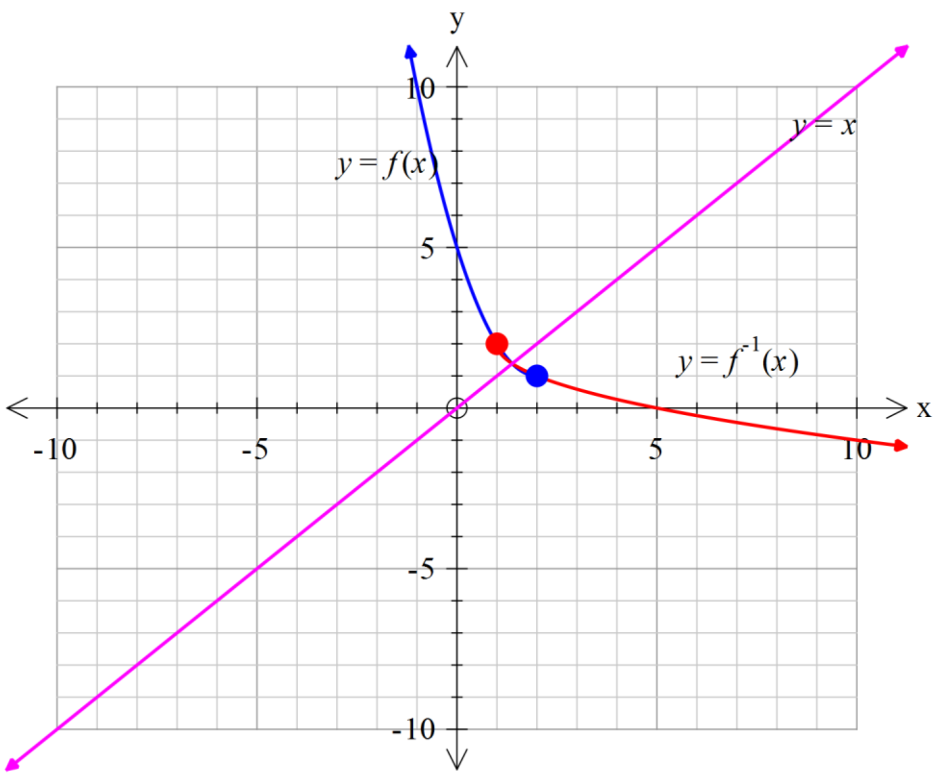
(b) If we restrict the domain of f to $x \le 2$, determine f^{-1}. (3 marks)

Solution
$y = x^2 - 4x + 5 = (x-2)^2 + 1, \quad x \le 2$ $x = (y-2)^2 + 1 \quad y \le 2$ $(y-2)^2 = x - 1$ $y - 2 = -\sqrt{x-1}$ since $y - 2 \le 0$ $y = 2 - \sqrt{x-1}$ $f^{-1}(x) = 2 - \sqrt{x-1}$
Specific behaviours
✓ interchanges x and y variables ✓ uses a negative in front of square root ✓ obtains an expression for f^{-1}

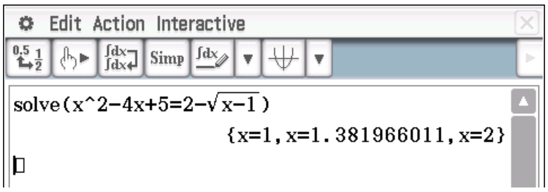
(c) Determine the domain and range of f^{-1}. (2 marks)

Solution
Domain $\{x : x \geq 1, x \in \mathbb{R}\}$ Range $\{y : y \leq 2, y \in \mathbb{R}\}$
Specific behaviours
✓ states domain ✓ states range

(d) Sketch f and f^{-1} from part (b) on the axes below. (4 marks)

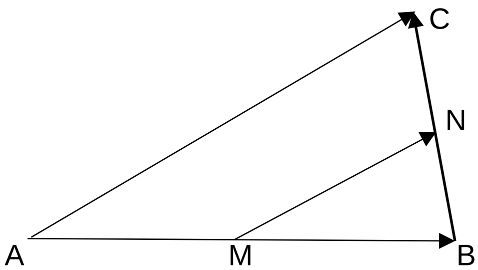
Solution

Specific behaviours
✓ f sketched for $x \leq 2$ ✓ graph of f has correct vertical intercept ✓ f^{-1} sketched for $x \geq 1$ ✓ clear that both are reflections of each other in line $y = x$

(e) Solve to one decimal place $f(x) = f^{-1}(x)$ showing these points on the graph in part (d). Comment on these points. (3 marks)

Solution
$x^2 - 4x + 5 = 2 - \sqrt{x-1} \Rightarrow \quad x = 1, x \approx 1.4, x = 2$  Points are $(1,2)$, $(1.4,1.4)$, $(2,1)$
Specific behaviours
✓ solves $f(x) = f^{-1}(x)$ ✓ determines the coordinates of the three points ✓ shows the three points clearly on the graph

Question 17 (5 marks)

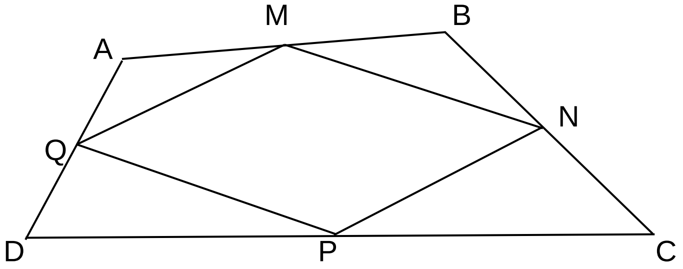
In ΔABC M is the midpoint of the side AB and N is the midpoint of the side BC.



(a) Show that $\overrightarrow{\boldsymbol{MN}} = \frac{1}{2}\overrightarrow{\boldsymbol{AC}}$. (2 marks)

Solution
$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \frac{1}{2}\overrightarrow{AC}$
Specific behaviours
✓ obtains second equality ✓ obtains third equality

(b) Deduce that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram. (3 marks)



Solution
By part (a) $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC} = \overrightarrow{QP}$ so the sides *MN* and *QP* are equal in length and parallel.
Similarly, $\overrightarrow{MQ} = \frac{1}{2}\overrightarrow{BD} = \overrightarrow{NP}$ so the sides *MQ* and *NP* are equal in length and parallel.
So *MNPQ* is a parallelogram
Specific behaviours
✓ deduces equality of length and parallelism of *MN* and *QP*
✓ same for *MQ* and *NP*
✓ draws correct conclusion

Question 18 (7 marks)

At a given time, t (=0) hours, a petri dish contains 0.2 grams of a particular bacteria. Sometime, t hours later, the bacteria had increased to an amount N (measured in grams). The rate of increase of the bacteria can be modelled by the logistical equation:

$$\frac{dN}{dt} = 2N - 3N^2$$

The general solution to the logistical equation is given by $N = \dfrac{2}{3 + Ce^{-2t}}$.

(a) Determine the value of the constant C. (1 mark)

Solution
Initially, (at $t = 0$) $0.2 = \dfrac{2}{3 + C}$ $\Rightarrow C = 7$
Specific behaviours
✓ determines C correctly

(b) Determine the limiting value of N. (1 mark)

Solution
$N(t \to \infty) = \dfrac{2}{3+0} = \dfrac{2}{3}$
Specific behaviours
✓ determines the correct limiting value

(c) Given $\dfrac{1}{N(2-3N)} = \dfrac{A}{N} + \dfrac{B}{2-3N}$, determine the values of the constants A and B.

(2 marks)

Solution
$\dfrac{1}{N(2-3N)} = \dfrac{A}{N} + \dfrac{B}{2-3N}$ $\Rightarrow 1 = A(2-3N) + BN$ At $N = 0, 1 = 2A \Rightarrow A = \dfrac{1}{2}$ At $N = \dfrac{2}{3}, 1 = B \times \dfrac{2}{3} \Rightarrow B = \dfrac{3}{2}$
Specific behaviours
✓ determines A correctly ✓ determines B correctly

(d) Using the rate of change equation and solving by separation of variables and partial fractions, show how to derive the general solution for N. (3 marks)

Solution
$\frac{dN}{dt} = 2N - 3N^2 = N(2-3N)$ $\therefore \int \frac{dN}{N(2-3N)} = \int dt$ $\Rightarrow \int \left(\frac{\frac{1}{2}}{N} + \frac{\frac{3}{2}}{(2-3N)} \right) dN = t + c \Rightarrow \frac{1}{2}\int \frac{dN}{N} + \frac{1}{2}\int \frac{3dN}{2-3N} = t + c$ $\Rightarrow \frac{1}{2}\left(\ln N - \ln(2-3N)\right) = t + c$ $\therefore \ln \frac{N}{2-3N} = e^{2t+c} = Ae^{2t} \quad (note: (2-3N) > 0)$ $\Rightarrow N = Ae^{2t}(2-3N) = 2Ae^{2t} - 3N \times Ae^{2t}$ $\therefore N = \frac{2Ae^{2t}}{(1+3Ae^{2t})} = \frac{2}{\frac{1}{A}e^{-2t} + 3} = \frac{2}{Ce^{-2t} + 3}$
Specific behaviours
✓ Integrates partial fractions using natural logs   ✓ expresses N as an exponential relationship with t ✓ rearranges to determine general solution

Question 19 (10 marks)

The displacement x m and the velocity v m/sec of a particle moving along a straight line are related according to the equation

$$9x^2 + 16v^2 = 25.$$

(a) Show that $16a = -9x$, where a cm/sec^2 is the acceleration. (2 marks)

Solution
$a = \frac{1}{2}\frac{d}{dx}(v^2) = \frac{1}{2}\frac{d}{dx}\left(\frac{25-9x^2}{16}\right) = -\frac{9}{16}x$
Specific behaviours
✓ uses the formula $a = \frac{1}{2}\frac{d}{dx}(v^2)$ ✓ differentiates and rearranges

(b) Determine the amplitude and period of this simple harmonic motion. (3 marks)

Solution
general formula for SHM $x(t) = A\sin(kt + \alpha)$ Since $9x^2 + 16v^2 = 25$, $A^2 = \frac{25}{9}$, and so $A = \frac{5}{3}$ i.e. the amplitude is $\frac{5}{3}\ m$ $a = -\frac{9}{16}x = -k^2x \Longrightarrow k = \frac{3}{4}$ So the period is $\frac{2\pi}{k} = \frac{8\pi}{3}$ sec.
Specific behaviours
✓ obtains amplitude ✓ uses $a = -k^2x = -\frac{9}{16}x$ to determine k ✓ obtains period

(c) Determine the first time at which the displacement of the particle is a maximum, given that initially the velocity of the particle is 1 cm/sec and this is increasing. (5 marks)

Solution
Since $9x^2 + 16v^2 = 25$ and $v(0) = 1$, $9x(0)^2 + 16 = 25$ and so $x(0) = \pm 1$ Since $a = -\frac{9}{16}x$ and $a(0) > 0 \Longrightarrow x(0) < 0 \Longrightarrow x(0) = -1$ Now $x(t) = \frac{5}{3}\sin(\frac{3}{4}t + \alpha)$ and so $x(0) = \frac{5}{3}\sin\alpha = -1$ So $\alpha = -\sin^{-1} 0.6 = -0.6435$ (to 4 decimal places) For maximum displacement $\frac{3}{4}t + \alpha = \frac{\pi}{2}$, i.e. $t = \frac{2}{3}\pi - \frac{4}{3}\alpha = 2.952$ (to 3 decimal places) So the time required to reach maximum displacement is 2.952 seconds
Specific behaviours
✓ obtains $x(0)^2 = 1$ ✓ obtains $x(0) = -1$ ✓ calculates α ✓ uses $\frac{3}{4}t + \alpha = \frac{\pi}{2}$ ✓ obtains correct answer

Question 20 **(9 marks)**

The height $h(t)$ metres of a hot-air balloon t seconds after take-off satisfies

$$\frac{dh}{dt} = \frac{500}{h+200} .$$

(a) How long does it take the balloon to rise 1 kilometre? (4 marks)

Solution
$\frac{dh}{dt} = \frac{500}{h+200} \Longrightarrow \int (h+200)dh = \int 500dt \Longrightarrow \frac{1}{2}(h+200)^2 = 500t + c$ $h = 0$ when $t = 0 \Longrightarrow c = \frac{1}{2} \times 200^2 = 20000$ So $(h+200)^2 = 1000t + 40000$ $h = 1000 \Longrightarrow 1200^2 = 1000t + 40000$, i.e. $t = 1400$ So it takes 1400 seconds for the balloon to rise 1 km.
Specific behaviours
✓ obtains $\int (h+200)dh = \int 500dt$ ✓ integrates both sides   ✓ determines the constant of integration   ✓ solves for t

The air temperature T °C at height h metres is given by

$$T = 300e^{-0.0001h} - 273 .$$

(b) Estimate the amount by which the temperature outside the balloon decreases in the 5 second period that starts when the balloon is 200 metres high. (5 marks)

Solution
$\delta T \approx \frac{dT}{dt}\delta t$ $\approx \frac{dT}{dh}\frac{dh}{dt}\delta t$ $\approx -0.03e^{-0.0001h}\frac{500}{h+200}\delta t$ When $h = 200$ and $\delta t=5$, $\delta T \approx -0.03e^{-0.02}\frac{5}{4}\delta t$ ≈ -0.184 So the temperature decreases by approximately 0.184°C in the 5-second time period.
Specific behaviours
✓ uses increments formula   ✓ uses chain rule   ✓ determines $\frac{dT}{dh}$ ✓ substitutes $h = 200$, $t = 5$ ✓ obtains correct answer

